

SHIFTED DUAL EQUIVALENCE AND SCHUR P -POSITIVITY

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ABSTRACT. By considering type B analogs of permutations and tableaux, we extend abstract dual equivalence to type B in two directions. In one direction, we define involutions on signed permutations and shifted tableaux that give a weak dual equivalence, thereby giving another proof of the Schur positivity of Schur Q - and P -functions. In another direction, we define an abstract shifted dual equivalence parallel to dual equivalence and prove that it can be used to establish Schur P -positivity of a function expressed as a sum of shifted fundamental quasisymmetric functions.

1. INTRODUCTION

Symmetric function theory can be harnessed by other areas of mathematics to answer fundamental enumerative questions. For example, multiplicities of irreducible components, dimensions of algebraic varieties, and various other algebraic constructions that require the computation of certain integers may often be translated to the computation of the coefficients of a given function in a particular basis. Often the chosen basis is the Schur functions, which arise as irreducible characters of representations of the general linear group, Frobenius characters of the symmetric group, and Schubert polynomials associated to the Grassmannian. Thus a quintessential problem in symmetric functions is to prove that a given function has nonnegative integer coefficients when expressed as a sum of Schur functions.

In [Assb], the author introduced dual equivalence graphs as a universal tool by which one can approach such problems. This tool has been applied to various important classes of symmetric functions, include LLT and Macdonald polynomials [Assb], k -Schur functions [AB12], and certain products of Schubert polynomials.

In this paper, we give a further application of dual equivalence to Schur Q - and P -functions. These functions arise in the study of projective representation of the symmetric group [Ste89], and have many nice properties parallel to Schur functions. In particular, they form dual bases for an important subspace of symmetric functions [Mac95]. While they have long been known to be Schur positive [Sag87], the application we provide lays the foundation for a stronger extension of dual equivalence to type B. We further define an abstract notion of shifted dual equivalence that offers a tool by which one can show that a given function has nonnegative coefficients when expanded in terms of Schur P -functions.

This paper is organized as follows. In Section 2, we introduce the classic combinatorial objects and their type B analogs. We use the combinatorics to make the connection with symmetric and quasisymmetric functions in Section 3. In Section 4, we review abstract dual equivalence and give an application to type B combinatorial objects in Section 5. Finally, in Section 6, we generalize the definitions and theorems of dual equivalence to the type B setting to define an abstract notion of shifted dual equivalence. Our main result, Theorem 6.5, is that this provides a universal tool for establishing Schur P -positivity.

2. PARTITIONS AND TABLEAUX

A *partition* λ is a non-increasing sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$. A partition is *strict* if its sequence of parts is strictly decreasing, i.e. $\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0$. The *size* of a partition is the sum of its parts, i.e. $\lambda_1 + \lambda_2 + \dots + \lambda_\ell$.

We identify a partition λ with its Young diagram, the collection of left-justified cells with λ_i cells in row i . For a strict partition, the shifted Young diagram is the Young diagram with row i shifted $\ell(\lambda) - i$ cells to the left. For examples, see Figure 1.

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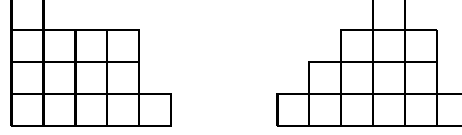


FIGURE 1. The Young diagram for $(5, 4, 4, 1)$ and the shifted Young diagram for $(6, 4, 3, 1)$.

A *semi-standard Young tableau* of shape λ is a filling of the Young diagram for λ with positive integers such that entries weakly increase along rows and strictly increase up columns. For example, see Figure 2.

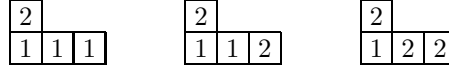


FIGURE 2. The set SSYT semi-standard Young tableaux of shape $(3, 1)$ with entries in $\{1, 2\}$.

A *semi-standard shifted tableau* of shape λ is a filling of the shifted Young diagram for λ with positive and negative integers such that entries weakly increase along rows and columns according to the ordering $1' < 1 < 2' < 2 < \dots$, each row at has most one entry $-i$ for each i and each column has at most one entry i for each i . For example, see Figure 3.



FIGURE 3. The semi-standard shifted tableaux of shape $(3, 1)$ with entries in $\{1', 1, 2', 2\}$ and no signed entries on the main diagonal.

The *reading word* of a semi-standard Young tableau or a semi-standard shifted tableau T , denoted $w(T)$, is the word obtained by reading the rows of T left to right, from top to bottom. For example, the reading words for the tableaux in Figure 2 from left to right are 2111, 2112, 2122, and the reading words for the tableaux in Figure 2 from left to right are 2111, 2112, 2112', 212'2.

A *permutation* of n is an ordering of the numbers $\{1, 2, \dots, n\}$. A *signed permutation* of n is a permutation of n together with a choice of sign for each $i = 1, 2, \dots, n$. A semi-standard Young or shifted tableau T is *standard* if its reading word is a permutation. For example, see Figures 4 and 5.

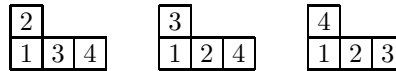


FIGURE 4. The standard Young tableaux of shape $(3, 1)$.

A semi-standard shifted tableau is *signed standard* if its reading word is a signed permutation. For example, each of the two standard shifted tableaux in Figure 5 has 16 signed analogs of which 4 have no signs on the main diagonal.

The *descent set* of a (signed) permutation is the subset of $\{1, 2, \dots, n-1\}$ given by

$$(2.1) \quad \text{Des}(w) = \left\{ |i| \mid \begin{array}{l} i \text{ unsigned and right of } i+1 \\ \text{or } i+1 \text{ signed and right } i \end{array} \right\}.$$

When we wish to emphasize n , we write Des_n .

in fact, symmetric functions. The Schur P - and Q -functions form dual bases for a subspace of symmetric functions spanned by the odd power sums, which, in degree n , has dimension equal to the number of strict partitions of n .

In particular, since the P - and Q -functions are symmetric, they can be expanded in the Schur basis. Stanley conjectured that the Schur coefficients in this expansion are nonnegative integers. This result follows as a corollary to Sagan's shifted insertion [Sag87].

Theorem 3.1 ([Sag87]). *For λ a strict partition of n , we have*

$$(3.4) \quad P_\lambda(X) = \sum_{\mu} a_{\mu} s_{\mu}(X)$$

where the sum is over all partitions μ of size n and a_{μ} are nonnegative integers.

One of the main results of this paper is to give another combinatorial proof of Theorem 3.1 using dual equivalence.

A basis for the space of quasisymmetric functions homogeneous of degree n is naturally indexed by subsets of $\{1, 2, \dots, n-1\}$. Gessel's fundamental basis for quasisymmetric functions [Ges84] is given by

$$(3.5) \quad F_D(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin D}} x_{i_1} \cdots x_{i_n}.$$

One great advantage of quasisymmetric functions is that they facilitate the use of standard in place of semi-standard objects. For example, we have the following expansion for Schur functions due to Gessel [Ges84].

Proposition 3.2 ([Ges84]). *For λ a partition of n , we have*

$$(3.6) \quad s_{\lambda}(X) = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des}(T)}(X),$$

where $\text{SYT}(\lambda)$ denotes the set of all standard Young tableaux of shape λ .

For example, for $n \leq 4$, the Schur expansions are

$$\begin{aligned} s_{(4)} &= F_{\{\emptyset\}} \\ s_{(3,1)} &= F_{\{1\}} + F_{\{2\}} + F_{\{3\}} \\ s_{(2,2)} &= F_{\{1,3\}} + F_{\{2\}} \\ s_{(2,1,1)} &= F_{\{1,2\}} + F_{\{1,3\}} + F_{\{2,3\}} \\ s_{(1,1,1,1)} &= F_{\{1,2,3\}} \end{aligned}$$

Analogously, we may express the Schur P functions in terms of the fundamental basis.

Proposition 3.3. *For λ a strict partition of n , we have*

$$(3.7) \quad P_{\lambda}(X) = \sum_{S \in \text{ShSYT}^{\pm}(\lambda)} F_{\text{Des}(S)}(X),$$

where $\text{ShSYT}^{\pm}(\lambda)$ denotes the set of all signed standard tableaux of shape λ .

Proof. The formula follows from (3.3) by standardizing the reading word while maintaining the positions of the signs. \square

For example, signing the entries in Figure 5, in all possible ways gives

$$P_{(3,1)} = F_{\{1\}} + 2F_{\{2\}} + F_{\{3\}} + F_{\{1,2\}} + 2F_{\{1,3\}} + F_{\{2,3\}}.$$

The similarity between (3.6) and (3.7) is the key to our proof of Theorem 3.1. In particular, it is easy to deduce from the above expansion that

$$P_{(3,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}.$$

However, notice that the summation in (3.7) is not over *standard* objects. For this, we need a new family quasisymmetric functions.

For $P \subseteq \{2, 3, \dots, n-1\}$, define the shifted fundamental quasisymmetric function $G_P(X)$ by

$$(3.8) \quad G_P(X) = \sum_{P \subseteq \text{Spike}(D)} F_D(X),$$

where the sum is over all subsets $D \subseteq \{1, 2, \dots, n-1\}$ for which $\text{Spike}(D)$ contains P . For example, for $n = 4$, we have

$$\begin{aligned} G_{\{1\}}(X) &= F_{\{1\}}(X) + F_{\{2\}}(X) + F_{\{1,3\}}(X) + F_{\{2,3\}}(X) \\ G_{\{2\}}(X) &= F_{\{2\}}(X) + F_{\{3\}}(X) + F_{\{1,3\}}(X) + F_{\{2,3\}}(X) \end{aligned}$$

The shifted fundamental quasisymmetric functions degree n form a basis for a subspace of quasisymmetric functions of degree n of dimension the n th Fibonacci number. The shifted fundamental quasisymmetric functions allow us to rewrite the Schur P -functions as follows.

Proposition 3.4. *For λ a strict partition of n , we have*

$$(3.9) \quad P_\lambda(X) = \sum_{S \in \text{ShSYT}(\lambda)} G_{\text{Peak}(S)}(X),$$

where $\text{ShSYT}(\lambda)$ denotes the set of all standard shifted tableaux of shape λ .

Proof. For T a standard shifted tableau. Then $i \in \text{Peak}(T)$ if and only if $i-1, i, i+1$ occur in $w(T)$ in the order $(i-1)(i+1)(i)$ or $(i+1)(i-1)(i)$. For S a signed standard tableau for which removing signs results in T , if i is signed, then $i-1 \in \text{Des}(S)$ and $i \notin \text{Des}(S)$, and if i is not signed, then $i-1 \notin \text{Des}(S)$ and $i \in \text{Des}(S)$. Therefore $\text{Peak}(T) \subseteq \text{Spike}(S)$. Furthermore, given any $D \supseteq \text{Peak}(T)$, there exists a signed standard tableau S with $|S| = T$ such that $\text{Spike}(S) = D$. The formula now follows from (3.7) and (3.8). \square

For example, using Figure 5, we compute

$$P_{(3,1)} = G_{\{1\}} + G_{\{2\}}.$$

4. DUAL EQUIVALENCE AND SCHUR POSITIVITY

Haiman [Hai92] defined *elementary dual equivalence involutions* on permutations as follows. If a, b are two consecutive letters of the word w , and c is also consecutive with a, b and appears between a and b in w , then interchanging a and b is an elementary dual equivalence move. When $\{a, b, c\} = \{i-1, i, i+1\}$, we denote this involution by d_i , and we regard words with c not between a and b as fixed points for d_i . For examples, see Figure 6.

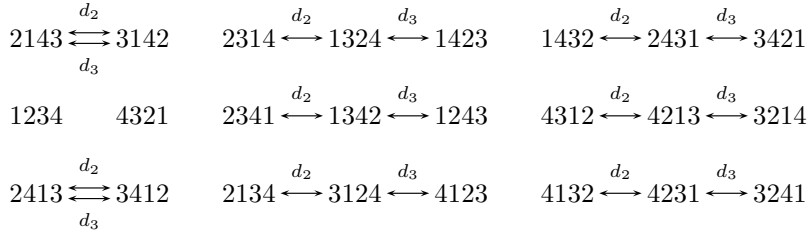


FIGURE 6. The dual equivalence classes of permutations of length 4.

Two permutations w and u are *dual equivalent* if there exists a sequence i_1, \dots, i_k such that $u = d_{i_k} \cdots d_{i_1}(w)$. Haiman [Hai92] showed that the dual equivalence involutions extend to standard Young tableaux via their reading words and that dual equivalence classes correspond precisely to all standard Young tableaux of a given shape, e.g. see Figure 7.

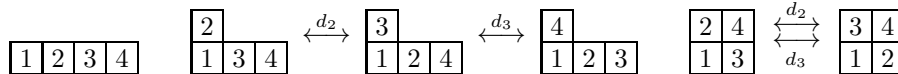


FIGURE 7. Three dual equivalence classes of SYT of size 4.

Given this, we may rewrite (3.6) in terms of dual equivalence classes as

$$(4.1) \quad s_\lambda(X) = \sum_{T \in [T_\lambda]} F_{\text{Des}(T)}(X),$$

where $[T_\lambda]$ denotes the dual equivalence class of some fixed $T_\lambda \in \text{SYT}(\lambda)$.

This paradigm shift to summing over objects in a dual equivalence class is the basis for the universal method for proving that a quasisymmetric generating function is symmetric and Schur positive. Motivated by (4.1), we have the abstract notion of dual equivalence for any set of objects endowed with a descent set.

Given $(\mathcal{A}, \text{Des})$ and involutions $\varphi_2, \dots, \varphi_{n-1}$, for $1 < j < i < n$ we consider the restricted dual equivalence classes $[T]_{(j,i)}$ generated by $\varphi_j, \dots, \varphi_i$. In addition, we consider the restricted and shifted descent sets $\text{Des}_{(j,i)}(T)$ obtained by intersecting $\text{Des}(T)$ with $\{j-1, \dots, i\}$ and subtracting $j-2$ from each element so that $\text{Des}_{(j,i)}(T) \subseteq [i-j+2]$.

Definition 4.1 ([Assa]). Let \mathcal{A} be a finite set, and let Des be a descent set map on \mathcal{A} such that $\text{Des}(T) \subseteq \{1, 2, \dots, n-1\}$ for all $T \in \mathcal{A}$. A (strong) dual equivalence for $(\mathcal{A}, \text{Des})$ is a family of involutions $\{\varphi_i\}_{1 < i < n}$ on \mathcal{A} satisfying the following conditions:

(i) The fixed points of φ_i are given by $\mathcal{A}^{\varphi_i} = \{T \in \mathcal{A} \mid i-1 \in \text{Des}(T) \Leftrightarrow i \in \text{Des}(T)\}$.

(ii) For $T \in \mathcal{A} \setminus \mathcal{A}^{\varphi_i}$, we have

$$\begin{aligned} h \in \text{Des}(T) &\Leftrightarrow h \in \text{Des}(\varphi_i(T)) && \text{for } h < i-2 \text{ or } i+1 < h, \\ i-2 \in \text{Des}(T) &\Leftrightarrow i-2 \notin \text{Des}(\varphi_i(T)) && \text{only if } T \notin \mathcal{A}^{\varphi_{i-1}}, \\ i+1 \in \text{Des}(T) &\Leftrightarrow i+1 \notin \text{Des}(\varphi_i(T)) && \text{only if } T \notin \mathcal{A}^{\varphi_{i+1}}, \\ j \in \text{Des}(T) &\Leftrightarrow j \notin \text{Des}(\varphi_i(T)) && \text{for } j = i-1 \text{ or } j = i. \end{aligned}$$

(iii) For $T \in \mathcal{A}$ and $|i-j| \geq 3$, we have $\varphi_j \varphi_i(T) = \varphi_i \varphi_j(T)$.

(iv) For all $1 \leq i-j \leq 3$ and all $T \in \mathcal{A}$ there exists a partition λ such that

$$\sum_{U \in [T]_{(j,i)}} F_{\text{Des}_{(j,i)}(U)}(X) = s_\lambda(X)$$

Dual equivalence was originally defined in [Assb] as a graph with signed vertices (given by the map Des) and colored edges (given by φ_i). The above reformulation of dual equivalence in terms of involutions can be found in [Assa].

By (4.1), dual equivalence classes of tableaux precisely correspond to Schur functions. Definition 4.1 was formulated so that the same property holds true for dual equivalence classes for any pair $(\mathcal{A}, \text{Des})$.

Theorem 4.2 ([Assa]). If $\{\varphi_i\}$ is a dual equivalence for $(\mathcal{A}, \text{Des})$, then

$$(4.2) \quad \sum_{T \in \mathcal{C}} F_{\text{Des}(T)}(X) = s_\lambda(X)$$

for any dual equivalence class \mathcal{C} and some partition λ of n . In particular, the quasisymmetric generating function for \mathcal{A} is symmetric and Schur positive.

One of our goals is to construct dual equivalence involutions for the set $\text{ShSYT}^\pm(\lambda)$ of signed standard shifted tableaux of shifted shape λ with descent function given by Des . Along the way, we define involutions on all signed permutations that satisfy a slightly weaker condition.

Definition 4.3 ([Assa]). Let \mathcal{A} be a finite set, and let Des be a descent set map on \mathcal{A} . A weak dual equivalence for $(\mathcal{A}, \text{Des})$ is a family of involutions $\{\varphi_i\}_{1 < i < n}$ on \mathcal{A} satisfying conditions (i), (ii), and (iii) of Definition 4.1 and the following:

(iv-a) For any $T \in \mathcal{A}$, the restricted generating function $\sum_{U \in [T]_{(i-1,i)}} F_{\text{Des}_{(i-1,i)}(U)}(X)$ is Schur positive.

Moreover, if $T, \varphi_i(T) \notin \mathcal{A}^{\varphi_{i-1}} \cup \mathcal{A}^{\varphi_i} \cup \mathcal{A}^{\varphi_{i+1}}$, then

$$\{\text{Des}_{(i-1,i)}(U) \mid U \in [T]_{(i-1,i)}\} = \{\text{Des}_{(i,i+1)}(U) \mid U \in [T]_{(i,i+1)}\}.$$

- (iv-b) For any $T \in \mathcal{A}$, the restricted generating function $\sum_{U \in [T]_{(i-2,i)}} F_{\text{Des}_{(i-2,i)}(U)}(X)$ is Schur positive. Moreover, if $T, \varphi_i(T) \notin \mathcal{A}^{\varphi_{i+1}}$ and $\varphi_i \varphi_{i-2}(T) \in \mathcal{A}^{\varphi_{i+1}}$, then, setting $U = \varphi_i(T)$, for all $m \geq 1$ such that $(\varphi_i \varphi_{i-2})^{m-1}(U) \notin \mathcal{A}^{\varphi_{i-2}}$ and $\varphi_{i-2}(\varphi_i \varphi_{i-2})^{m-1}(U) \notin \mathcal{A}^{\varphi_i}$, we have $(\varphi_i \varphi_{i-2})^m(U) \notin \mathcal{A}^{\varphi_{i+1}}$.

Weak dual equivalence classes are larger than strong dual equivalence classes, but they still have the fundamental property of being Schur positive.

Theorem 4.4 ([Assa]). *If $\{\varphi_i\}$ is a weak dual equivalence for $(\mathcal{A}, \text{Des})$, then*

$$(4.3) \quad \sum_{T \in \mathcal{C}} F_{\text{Des}(T)}(X)$$

is symmetric and Schur positive for any weak dual equivalence class \mathcal{C} .

5. DUAL EQUIVALENCE FOR SIGNED PERMUTATIONS

The objects for which we will ultimately construct a dual equivalence are the signed standard tableaux with the associated descent function given by (2.2). We begin, however, by defining involutions on signed permutations with the descent set given by (2.1).

Definition 5.1. Let w be a signed permutation. For $1 < i < n$, let $\{a, b, c\} = \{i-1, i, i+1\}$ where a is left of b and b is left of c in w . Then $\varphi_i(w)$ is given by the following rule:

- if $i \notin \text{Spike}(w)$, then $\varphi_i(w) = w$;
- else if exactly one of b, c is signed, then $\varphi_i(w)$ swaps the signs of b, c ;
- else $\varphi_i(w)$ swaps a and c leaving the signs in the original positions.

Theorem 5.2. *The maps $\{\varphi_i\}_{1 < i < n}$ give a weak dual equivalence for signed permutations.*

Proof. We begin by showing that the maps φ_i are involutions. This is clearly the case for w for which $i \notin \text{Spike}(w)$, so suppose $i \in \text{Spike}(w)$. If exactly one of b, c is signed, then this also holds for $\varphi_i(w)$, so it suffices to show that $i \in \text{Peak}(w)$ if and only if $i \in \text{Spike}(\varphi_i(w)) \setminus \text{Peak}(\varphi_i(w))$. If $i \in \text{Peak}(w)$, then either $i-1$ precedes an unsigned i or i precedes a signed $i-1$, and either either $i+1$ precedes an unsigned i or i precedes a signed $i+1$. When exactly one of b, c is signed, this means 2 must be unsigned and $a \neq 2$. Checking the eight possible cases verifies the claim for this case. Similarly, when both or neither b, c is signed, we must have $b \neq 2$, and we again consider the eight possible cases to verify the claim. Therefore φ_i is indeed an involution that flips spikes. This also establishes (i) of Definition 4.1 as well as the last case of (ii). Since φ_i is completely determined by the relative positions of $\{i-1, i, i+1\}$, both (iii) and the first case of (ii) follow from the fact that $\{i-1, i, i+1\} \cap \{j-1, j, j+1\}$ are disjoint for $|i-j| \geq 3$. The middle two cases of (ii) can be resolved by considering the same sixteen cases of the previous argument. Finally, the local Schur positivity and conditions (iv-a) and (iv-b) of Definition 4.3 can be established by checking the $2^6 6! = 46,080$ cases for signed permutations of length 6. \square

We cannot directly extend the maps φ_i signed standard tableaux via reading words. For example, the shifted tableau of shape $(2, 1)$ with reading word 312 would map under φ_2 to a filling of $(2, 1)$ that violates the increasing columns rule. Therefore we present the following modification of φ_i for reading words of signed standard tableaux.

Definition 5.3. Let S be a signed standard tableau. For $1 < i < n$, let $\{a, b, c\} = \{i-1, i, i+1\}$ where a is left of b and b is left of c in w . Then $\psi_i(w)$ is given by the following rule:

- if $i \notin \text{Spike}(S)$, then $\psi_i(S) = S$;
- else if a, c are in the same column, then $\psi_i(S)$ toggles the sign on c ;
- else if exactly one of b, c is signed, then $\psi_i(S)$ swaps the signs of b, c ;
- else $\psi_i(S)$ swaps a and c leaving the signs in the original positions.

For examples of ψ on signed standard tableaux, see Figure 8.

Proposition 5.4. *The maps $\{\psi_i\}$ give well-defined involutions on the set of signed standard tableaux. Furthermore, if S has no signed entries along the main diagonal, then neither does $\psi_i(S)$ for any i .*

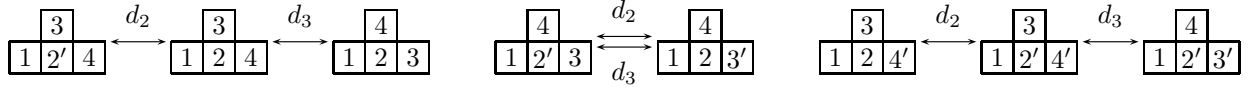


FIGURE 8. Dual equivalence for $\text{ShSYT}^\pm(3, 1)$.

Proof. By Theorem 5.2, it is enough to show that ψ_i preserves the spike at i whenever a and c are in the same column. For this case, we must have $a > c$ and b is either in the row of a and immediately to its right or in the row of c and immediately to its left. In the former case, there must be a cell directly below b , say containing $x \neq a$, with $c < x < b$ contradicting that $c < a < b$ are consecutive. Hence the latter case must hold, so $abc = (i+1)(i-1)(i)$. All eight possible signs result in a spike at i , and this spike is a peak if and only if $c = i$ is signed. Therefore ψ_i is indeed an involution on signed standard tableaux.

Furthermore, when a and c are in the same column, if there is a cell left of b , then there must be a cell directly above b , say containing $x \neq c$, with $b < x < a$, again contradicting that $b < c < a$ are consecutive. Therefore both a, b are the leftmost cells in their rows and c is not. In this case, ψ_i preserves the signs (or lack thereof) along the main diagonal. We claim that if a, c are not in the same column, then a appears weakly left of both b, c . Indeed, if b is left of a then since a comes earlier in the reading word than b , a must be in a strictly higher row than b . Therefore there is a cell right of and in the same row as b and below and in the same column as a , so that $b < x < a$. Since a, b, c are consecutive, we must have $x = c$, so a, c are in the same column. A similar argument shows that c must be strictly right of a . Since ψ_i never changes the sign of the position containing a , it necessarily preserves the signs along the main diagonal. \square

Theorem 5.5. *The maps $\{\psi_i\}_{1 \leq i \leq n}$ give a weak dual equivalence for signed standard tableaux. In particular, the Schur Q - and P -functions are Schur positive.*

Proof. Completely analogous to the proof of Theorem 5.2. Proposition 5.4 establishes that the maps are involutions as well as (i) of Definition 4.1 and the last case of (ii). Since ψ_i is completely determined by the relative positions of $\{i-1, i, i+1\}$, both (iii) and the first case of (ii) follow from the fact that $\{i-1, i, i+1\} \cap \{j-1, j, j+1\}$ are disjoint for $|i-j| \geq 3$. The middle two cases of (ii) can be resolved by considering the sixteen cases when S has a peak at i . Finally, condition (iv) can be established by verifying $2^6 6!$ cases with each possible assumption for whether two given positions appear in the same column. \square

6. SHIFTED DUAL EQUIVALENCE GRAPHS

Haiman [Hai92] also defined *elementary shifted dual equivalence involutions* on (unsigned) permutations that extends to standard shifted tableaux. If a, b are two consecutive letters of the word w , c is also consecutive with a, b and appears between a and b in w , and d is also consecutive with a, b, c and appears left of c in w , then interchanging a and b is an elementary shifted dual equivalence move. When $\{a, b, c, d\} = \{i-1, i, i+1, i+2\}$, we denote this involution by b_i , and we regard words with c not between a and b or with d right of c as fixed points for b_i . This rule is illustrated in Figure 9.

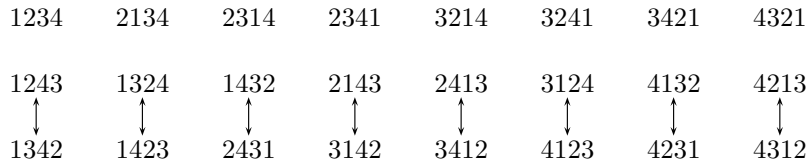


FIGURE 9. The shifted dual equivalence classes of permutations of length 4.

Haiman [Hai92] showed that the shifted dual equivalence involutions extend to standard shifted tableaux via their reading words and that dual equivalence classes correspond precisely to all standard shifted tableaux of a given shape. For examples, see Figures 10 and 11.

Comparing Figure 10 with Figure 7, it might seem that shifted dual equivalence classes are the same as dual equivalence classes. However, shifted classes can have triple edges, whereas dual equivalence classes can

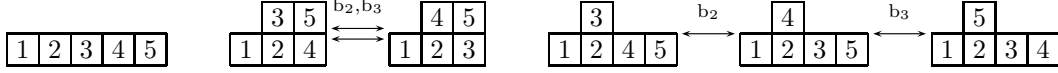


FIGURE 10. The shifted dual equivalence classes of ShSYT of size 5.

have at most double edges, so the equality is an artifact of small numbers. To make this statement precise, we introduce the notion of an morphism between dual equivalences.

Definition 6.1. Let \mathcal{A}, \mathcal{B} be two sets of combinatorial objects, and let Des_A and Des_B be subset statistics on each. Given involutions α_i on \mathcal{A} and β_i on \mathcal{B} , a *morphism* from $(\mathcal{A}, \text{Des}_A, \alpha)$ to $(\mathcal{B}, \text{Des}_B, \beta)$ is a map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that for every $a \in \mathcal{A}$, we have $\text{Des}_A(a) = \text{Des}_B(\phi(a))$ and $\phi(\alpha_i(a)) = \beta_i(\phi(a))$. A morphism is an *isomorphism* if it is a bijection from \mathcal{A} to \mathcal{B} .

To avoid combersome notation, we omit the subscript for Des when it is clear from context.

Since dual equivalence pertains to descent sets and shifted dual equivalence pertains to peak sets, we also need to shift a peak set to a subset as follows. Given a subset D of positive integers greater than 1, let $D - 1$ be the subset obtained by subtracting 1 from each element of D .

Proposition 6.2. For nonnegative integers $r > s$, the shifted dual equivalence for $(\text{ShSYT}((r, s)), \text{Peak} - 1)$ given by $\{b_i\}$ is isomorphic to the dual equivalence on $(\text{SYT}((r - 1, s)), \text{Des})$ given by $\{d_i\}$. For λ a strict partition with more than 2 parts, the shifted dual equivalence on $(\text{ShSYT}(\lambda), \text{Peak} - 1)$ given by $\{b_i\}$ is not isomorphic to $(\text{SYT}(\mu), \text{Des})$ given by $\{d_i\}$ for any partition μ .

Proof. Consider the map ϕ from $\text{ShSYT}((r, s))$ to $\text{SYT}((r - 1, s))$ given by removing the cell containing 1, subtracting 1 from each entry. On the level of sets, ϕ is clearly a bijection. One easily checks that, in addition, $\text{Peak}(T) - 1 = \text{Des}(\phi(T))$, and $\phi(b_{i+1}(T)) = d_i(\phi(T))$. Therefore ϕ is an isomorphism of dual equivalences.

The shifted tableau T of shape $\lambda = (3, 2, 1)$ with reading word 645123 has $b_2 = b_3 = b_4$. Any strict partition with at least 3 parts must contain λ , and so it contains an element that restricts to T . In particular, such an element has $b_2 = b_3 = b_4$, and so the equivalence cannot be isomorphic to any dual equivalence by condition (ii) of Definition 4.1. \square

Completely analogous to the unshifted case, for λ a strict partition, we may rewrite (3.9) in terms of dual equivalence classes as

$$(6.1) \quad P_\lambda(X) = \sum_{T \in [T_\lambda]} G_{\text{Peak}(T)}(X),$$

where $[T_\lambda]$ denotes the shifted dual equivalence class of some fixed $T_\lambda \in \text{ShSYT}(\lambda)$.

By (6.1), shifted dual equivalence classes of standard shifted tableaux precisely correspond to Schur P -functions. Following the analogy, our goal is to use this paradigm shift to summing over objects in a shifted dual equivalence class to give a universal method for proving that a quasisymmetric generating function is symmetric and Schur P -positive.

Since the subset statistic for this case is the peak set instead of the descent set, we make the following notation. Given $(\mathcal{A}, \text{Peak})$ and involutions $\varphi_2, \dots, \varphi_{n-2}$, for $1 < j < i < n - 1$ we consider the restricted shifted dual equivalence classes $[T]_{(j,i)}$ generated by $\varphi_j, \dots, \varphi_i$. In addition, we consider the restricted peak sets $\text{Peak}_{(j,i)}(T)$ obtained by intersecting $\text{Peak}(T)$ with $\{j - 1, \dots, i + 1\}$ and subtracting $j - 2$ from each element so that $\text{Peak}_{(j,i)}(T) \subseteq [i - j + 1]$.

Definition 6.3. Let \mathcal{A} be a finite set, and let Peak be a peak map on \mathcal{A} such that $\text{Peak}(T) \subseteq \{2, \dots, n - 1\}$ with no consecutive entries for all $T \in \mathcal{A}$. A (strong) shifted dual equivalence for $(\mathcal{A}, \text{Peak})$ is a family of involutions $\{\varphi_i\}_{1 < i < n - 1}$ on \mathcal{A} satisfying the following conditions:

(i) The fixed points of φ_i are given by $\mathcal{A}^{\varphi_i} = \{T \in \mathcal{A} \mid i, i + 1 \notin \text{Peak}(T)\}$.

(ii) For $T \in \mathcal{A} \setminus \mathcal{A}^{\varphi_i}$, we have

$$\begin{aligned} i \in \text{Peak}(T) &\Leftrightarrow i + 1 \in \text{Peak}(\varphi_i(T)) \\ h \in \text{Peak}(T) &\Leftrightarrow h \in \text{Peak}(\varphi_i(T)) \quad \text{for } h \leq i - 2 \text{ and } i + 3 \leq h \end{aligned}$$

(iii) For $T \in \mathcal{A}$ and $|i - j| \geq 4$, we have $\varphi_j \varphi_i(T) = \varphi_i \varphi_j(T)$.

(iv) For all $1 \leq i - j \leq 4$ and all $T \in \mathcal{A}$ there exists a strict partition λ such that

$$\sum_{U \in [T]_{(j,i)}} G_{\text{Peak}_{(j,i)}(U)}(X) = P_\lambda(X).$$

Definition 6.3 completely characterizes shifted dual equivalence class in the same sense that Definition 4.1 characterizes dual equivalence classes.

Proposition 6.4. *For λ a strict partition of n , the involutions $\{b_i\}_{1 \leq i \leq n-1}$ give a shifted dual equivalence for $\text{ShSYT}(\lambda)$ with peak function given by (2.3).*

Proof. For condition (i) and the first line of condition (ii), it is enough to check the statements for permutations of length 4, which are given in Figure 9. For the second line of condition (ii), note that whether $h \in \text{Peak}$ depends only on the relative positions of $h-1, h, h+1$ and b_i only changes positions of $i-1, i, i+1, i+2$. Similarly, condition (iii) follows from the fact that b_i depends only on the relative positions of $i-1, i, i+1, i+2$ and for $|i - j| \geq 4$ these sets are disjoint. Condition (iv) holds by definition. \square

The real purpose of Definition 6.3 is to establish the following analog of Theorem 4.2.

Theorem 6.5. *If $\{\varphi_i\}_{1 \leq i \leq n-1}$ is a shifted dual equivalence for $(\mathcal{A}, \text{Des})$, then*

$$(6.2) \quad \sum_{T \in \mathcal{C}} G_{\text{Peak}(T)}(X) = P_\lambda(X)$$

for any shifted dual equivalence class \mathcal{C} and some strict partition λ of n . In particular, the quasisymmetric generating function for \mathcal{A} is symmetric and Schur P -positive.

We prove Theorem 6.5 along the same lines as the structure theorem for dual equivalence given in [Assb]. We present the appropriate analogues of statements from [Assb], omitting proofs when they are completely analogous.

Proposition 6.6. *For λ, μ strict partitions, if $\phi : \text{ShSYT}(\lambda) \rightarrow \text{ShSYT}(\mu)$ is an isomorphism of shifted dual equivalences, then $\lambda = \mu$ and $\phi = \text{id}$.*

The proof of Proposition 6.6 is completely analogous to the proof of Proposition 3.11 in [Assb].

Given strict partitions λ, μ with $\lambda \subset \mu$, fix a filling of the cells of $\mu \setminus \lambda$, say A . Let $\text{ShSYT}(\lambda, A) \subset \text{ShSYT}(\mu)$ be subset of shifted standard tableaux that restrict to A when skewed by λ . The resulting shifted dual equivalence on $\text{ShSYT}(\lambda, A)$ has the same involutions as $\text{ShSYT}(\lambda)$, but the Peak function has now been extended. With this in mind, we show that for λ a partition of n , any extension of the peak function for $\text{ShSYT}(\lambda)$ can be modeled by $\text{ShSYT}(\lambda, A)$ for some augmenting tableau A .

In general, when considering a shifted dual equivalence for $(\mathcal{A}, \text{Peak})$, we allow the peak function to give a subset of $\{2, 3, \dots, N-1\}$ for any $n \leq N$. To facilitate the case when $N > n$, we write $\text{Peak}_n = \text{Peak} \cap \{2, 3, \dots, n-1\}$ for the restriction of the peak set to the usual case.

Lemma 6.7. *Let $\{\varphi_i\}_{1 \leq i \leq n-1}$ be a shifted dual equivalence for $(\mathcal{A}, \text{Peak})$ such that there is a unique shifted dual equivalence class, and let ϕ be an isomorphism from the $(\mathcal{A}, \text{Peak}_n)$ to $(\text{ShSYT}(\lambda), \text{Peak})$ for some strict partition λ of n . Then there exists a standard shifted tableau A of shape μ/λ , μ a strict partition of size N , with entries $n+1, \dots, N$ such that ϕ gives an isomorphism from $(\mathcal{A}, \text{Peak})$ to $(\text{ShSYT}(\lambda, A), \text{Peak})$. Moreover, the position of the cell of A containing $n+1$ is unique.*

The proof of Lemma 6.7 is completely analogous to the proof of Lemma 3.13 in [Assb].

Lemma 6.8. *A family of involutions $\{\varphi_i\}$ is a shifted dual equivalence for $(\mathcal{A}, \text{Peak})$ if it satisfies conditions (i), (ii), and (iii) of Definition 6.3 the following*

- (v) *for every $T \in \mathcal{A}$ and $1 \leq i - j \leq 3$, there is an isomorphism from $([T]_{(j,i)}, \text{Peak}_{(j,i)})$ to $(\text{ShSYT}(\lambda), \text{Peak})$ for some strict partition λ .*
- (vi) *for every $T, S \in \mathcal{A}$, if $S \notin [T]_{(i-1,1)}$, then $[T]_{(i-4,i)}$ and $[S]_{(i-4,i)}$ are nonisomorphic.*

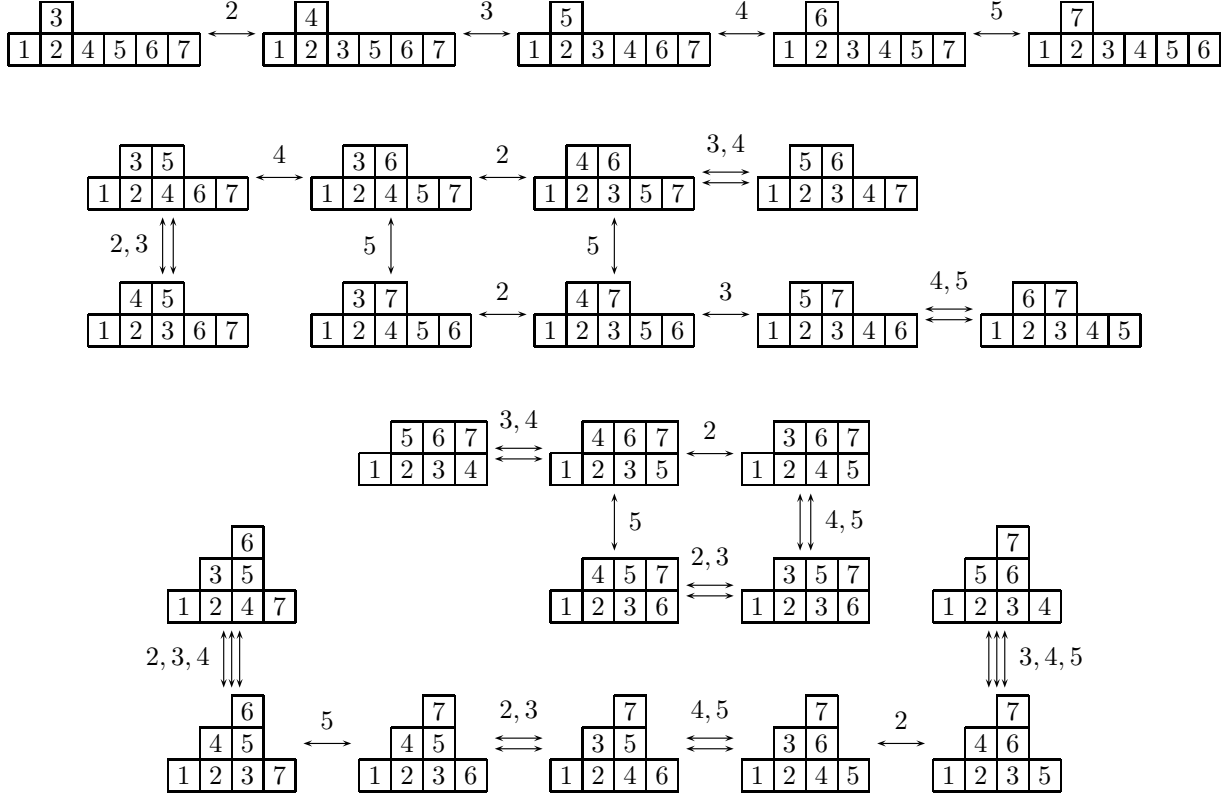


FIGURE 11. The standard shifted dual equivalences of size 7.

The proof of Lemma 6.8 is completely analogous to the proof of Theorem 5.1 in [Assa]. Its purpose is to facilitate a straightforward proof of the following result analogous to Theorem 3.14 in [Assb].

Theorem 6.9. *Let $\{\varphi_i\}_{1 \leq i \leq n}$ be a family of involutions on $(\mathcal{A}, \text{Peak})$, satisfying conditions (i), (ii), (iii) of Definition 6.3 and condition (v) of Lemma 6.8, and suppose there is a morphism from $(\mathcal{A}, \text{Peak}_n)$ under $\{\varphi_i\}_{1 \leq i \leq n-1}$ to $(\text{ShSYT}(\lambda), \text{Peak})$ under $\{b_i\}_{1 \leq i \leq n-1}$ for some strict partition λ of n . Then there exists a morphism ϕ from $(\mathcal{A}, \text{Peak})$ under $\{\varphi_i\}_{1 \leq i \leq n}$ to $(\text{ShSYT}(\mu), \text{Peak})$ under $\{b_i\}_{1 \leq i \leq n}$ for a unique strict partition μ of $n+1$.*

Corollary 6.10. *Let $\{\varphi_i\}_{1 \leq i \leq n}$ be a family of involutions on $(\mathcal{A}, \text{Peak})$ satisfying the hypotheses of Theorem 6.9. Then the fiber over each standard shifted tableaux in the morphism to $\text{ShSYT}(\mu)$ has the same cardinality.*

Completely analogous to the proof of Theorem 3.9 in [Assb], we obtain the following structure theorem for shifted dual equivalence by invoking condition (vi) of Lemma 6.8.

Theorem 6.11. *Given any shifted dual equivalence $\{\varphi_i\}_{1 \leq i \leq n-1}$ for $(\mathcal{A}, \text{Peak})$, there exists a unique strict partition λ of size n such that there is an isomorphism of shifted dual equivalences between $(\mathcal{A}, \text{Peak})$ under $\{\varphi_i\}_{1 \leq i \leq n-1}$ and $(\text{ShSYT}(\lambda), \text{Peak})$ under $\{b_i\}_{1 \leq i \leq n-1}$.*

Theorem 6.5 now follows.

We conclude with a vague conjecture that some analog of weak dual equivalence exists for the shifted case along similar lines. That is, weak shifted dual equivalence classes should be locally Schur P -positive with some extra conditions imposed on their structure. We await a compelling application to help guide the precise statement of this conjecture.

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